Energy estimates for regularly hyperbolic operators with non-classical symbols

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(Received September 30, 2011)

Synopsis

Energy estimates for regularly hyperbolic operators with $S^m_{1,2/3}$ class symbols are drawn.
KEYWORDS: energy estimates, regularly hyperbolic

1. Introduction

Consider the wave operator with an oscillating coefficient

$$L_\varepsilon u = \partial_t^2 u - a_\varepsilon^2(t)\partial_x^2 u = 0$$

with $a_\varepsilon(t) = a(t/\varepsilon)$. Here $a(t)$ is a positive and periodic smooth function and ε is a small positive parameter. For the standard energy $E(t) = \|u_t\|^2 + \|u_x\|^2$ of a solution u, we have

$$E(t) \le e^{C\frac{t}{\varepsilon}} E(0).$$

In order to improve the above estimate, we consider the decomposition of $\hat{L}_\varepsilon = \frac{d^2}{dt^2} + a_\varepsilon^2(t)\xi^2$:

$$(\frac{d}{dt} \pm ia_\varepsilon(t)\xi + a_0(t) \pm \frac{i}{\xi}a_1(t))(\frac{d}{dt} \mp ia_\varepsilon(t)\xi - a_0(t) \mp \frac{i}{\xi}a_1(t)) = \hat{L}_\varepsilon + R_\pm(t,\xi) \tag{1}$$

with $a_0(t) = \dfrac{-a_\varepsilon'(t)}{2a_\varepsilon(t)}$, $a_1(t) = \dfrac{1}{2a_\varepsilon(t)}(a_0'(t) + a_0(t)^2)$ and

$$R_\pm(t,\xi) = \mp\frac{i}{\xi}(2a_1(t)a_0(t) + a_1'(t)) + \frac{a_1(t)^2}{\xi^2}.$$

Then we see that

$$\hat{u}_\pm = a_\varepsilon^{-1/2}(t)(\frac{d}{dt} \mp ia_\varepsilon(t)\xi - a_0(t) \mp \frac{i}{\xi}a_1(t))\hat{u}$$

satisfies

$$(\frac{d}{dt} \pm ia_\varepsilon(t)\xi \pm \frac{i}{\xi}a_1(t))\hat{u}_\pm = a_\varepsilon^{-1/2}(t)\hat{L}_\varepsilon\hat{u} + R_\pm(t,\xi)\hat{u}_\pm. \tag{2}$$

We define the energy $e_1(t)$ by

$$e_1(t) = \frac{1}{2}(|\hat{u}_+|^2 + |\hat{u}_-|^2) = \frac{1}{a_\varepsilon}(|\frac{d}{dt}\hat{u} - a_0\hat{u}|^2 + |(a_\varepsilon\xi + \frac{a_1}{\xi})\hat{u}|^2.$$

Note that

$$|a_0(t)| \le \frac{C}{\varepsilon}, \quad |a_1(t)| \le \frac{C}{\varepsilon^2}, \quad \text{and } |R_\pm(t,\xi)| \le C(\frac{1}{|\xi|\varepsilon^3} + \frac{1}{|\xi|^2\varepsilon^4}).$$

Then, when $|\xi|\varepsilon$ is large, we see $C^{-1}e_1(t) \le (|\hat{u}_t|^2 + \xi^2|\hat{u}|^2) \le Ce_1(t)$ with some $C > 0$. Hence from (2) follows that a solution $\hat{u}$ of $\hat{L}_\varepsilon\hat{u} = 0$ satisfies, when $|\xi|\varepsilon$ is large,

$$\frac{d}{dt}e_1(t) \le \frac{C}{|\xi|^2\varepsilon^3}e_1(t),$$

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$$e_1(t) \le e^{\frac{Ct}{|\xi|^2\varepsilon^3}} e_1(0), \quad (t \ge 0).$$

Hence, the high frequency energy $E_\varepsilon(t) = \int_{|\xi|\ge C\varepsilon^{-3/2}} (|\hat{u}_t|^2 + \xi^2|\hat{u}|^2)\, d\xi$ with large C of a solution $L_\varepsilon u = 0$, has the estimate

$$E_\varepsilon(t) \le C_1 e^{C_2 t} E_\varepsilon(0) \quad (t \ge 0)$$

(see the related results [1),2)]).

In order to obtain a similar estimate for the regularly hyperbolic operator

$$P_\varepsilon u = D_t^m u + \sum_{1\le j\le m, |\alpha|\le j} a_{j,\alpha}(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}) D_t^{m-j} D_x^\alpha u,$$

we are led to study the property of the operator P_ε in the case where $\varepsilon|\xi|^{2/3}$ is large.

Noting, if $\varepsilon^{-1} \le C|\xi|^{2/3}$,

$$|\partial_t^k \partial_x^\alpha a_j(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})| \le C_1 \varepsilon^{-(k+|\alpha|)} \le C_2 |\xi|^{(k+|\alpha|)\frac{2}{3}},$$

we consider, in this paper, the regularly hyperbolic operator P whose symbol $p(t,x,\tau,\xi)$ is given by

$$p(t,x,\tau,\xi) = \tau^m + \sum_{j=1}^m a_j(t,x,\xi)\tau^{m-j}$$

with real $a_j(t,x,\xi)$ satisfying

$$|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a_j(t,x,\xi)| \le C\langle\xi\rangle^{j-|\beta|+(k+|\alpha|)\frac{2}{3}}$$

that is, a_j in $S^j_{1,2/3}$ where

$$\langle\xi\rangle = (1+|\xi|^2)^{1/2}.$$

More precisely, for given $T>0$, we denote by $S^k_{\rho,\delta}([0,T])$ the set of symbols $a(t,x,\xi) \in C^\infty([0,T]\times \mathbb{R}^n \times \mathbb{R}^n)$ satisfying for any non-negative integer l and any multi-index $\alpha, \beta \in \mathbb{N}^n$, where $\mathbb{N} = \{0,1,2,\dots\}$,

$$|\partial_t^l \partial_x^\alpha \partial_\xi^\beta a(t,x,\xi)| \le C_{l,\alpha,\beta}\langle\xi\rangle^{k-\rho|\beta|+\delta(l+|\alpha|)}$$

on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^n$.

Let P be an operator on $[0,T]\times\mathbb{R}^n$:

$$Pu = D_t^m u + \sum_{j=1}^m A_j(t,x,D_x) D_t^{m-j} u$$

whose symbol $p(t,x,\tau,\xi) = \tau^m + \sum_{j=1}^m a_j(t,x,\xi)\tau^{m-j}$ with real symbols $a_j(t,x,\xi) \in S^j_{1,2/3}([0,T])$ has a factorization : if $|\xi| \ge R_0$ with some $R_0 > 0$,

$$p(t,x,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,x,\xi)) \tag{3}$$

with real $\lambda_j(t,x,\xi)$ satisfying, with some $\delta > 0$,

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \ge \delta\langle\xi\rangle \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \text{ with } |\xi| \ge R_0 \tag{4}$$

for any $j,k \in I$ with $j \ne k$.

Here $I = \{1, 2, \dots, m\}$ and $D_t = \frac{1}{i}\partial_t$. For a symbol $a(t,x,\xi) \in S^j_{1,2/3}([0,T])$, we define the operator $A(t,x,D_x)$ by

$$A(t,x,D_x)u = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(t,x,\xi)u(y)\,dyd\xi$$

and for $q(t,x,\tau,\xi) = \sum_{j=0}^{l} a_j(t,x,\xi)\tau^{l-j}$

$$Q(t,x,D_t,D_x) = \sum_{j=0}^{l} A_j(t,x,D_x)D_t^{l-j}.$$

We show the following energy estimate for P.

Theorem 1. *We assume that the opretator P satisfies the above mentioned conditions* (3) *and* (4)*. Then we have the following:*

$$\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(t,\cdot)\|_{-(m-1)/2} \le$$
$$C(\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(0,\cdot)\|_{-(m-1)/2} + \int_0^t \|(P+P_{sb})u(s,\cdot)\|_{-(m-1)/2}\,ds) \quad (5)$$

for any $u(t,x) \in C^\infty([0,T]\times\mathbb{R}^n)$ that is rapidly decreasing with respect to x, where P_{sb} is the operator whose symbol $p_{sb}(t,x,\tau,\xi)$ is given by $p_{sb}(t,x,\tau,\xi) = \frac{1}{2i}\sum_{\mu=0}^{n}\partial_{x_\mu}\partial_{\xi_\mu}p(t,x,\tau,\xi)$ with $x_0 = t, \xi_0 = \tau$.

Here $D_{x_j} = \frac{1}{i}\partial_{x_j}$ and $D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$. The norm $\|\cdot\|_\sigma$ stands for the standard Sobolev norm given by $\|u(\cdot)\|_\sigma^2 = \int(1+|\xi|^2)^\sigma|\hat{u}(\xi)|^2\,d\xi$ with the Fourier transform $\hat{u}(\xi)$ of $u(x)$.

Remark 1. The differential operator whose principal symbol has only real roots satisfying (3) and (4) is called the regularly hyperbolic operator.

Remark 2. We remark that the above estimate (5) dose not necessarily imply the wellposedness of the Cauchy problem for $P + P_{sb}$. The application of the above estimate to the operator with oscillating coefficients will be discussed in a forthcoming paper.

If $m = 1$, Theorem 1 is evident. We assume $m \ge 2$ from now on. In order to draw the estimate (5), we use the decomposition of P similar to (1). For the simplicity of expression, we use the Weyl calculus of pseudodifferential operators. For the properties of such calculus, refer to the section 14 in Chapter 7 of Taylor's book[3]. For symbols $a(t,x,\xi)$ and $q = \sum_{j=1}^{l} a_j(t,x,\xi)\tau^{l-j}$, we define operators A^w and Q^w by

$$A^w(t,x,D_x)u = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a(t,\frac{x+y}{2},\xi)u(y)\,dyd\xi \quad (6)$$

and

$$Q^w(t,x,D_t,D_x)u = \sum_{j=0}^{l}\frac{1}{(2\pi)^{n+1}}\int e^{i((t-s)\tau+(x-y)\xi)}a_j(\frac{t+s}{2},\frac{x+y}{2},\xi)\tau^{l-j}u(s,y)\,dsdyd\tau d\xi$$

with some appropriately extended $a_j(t,x,\xi)$ on $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. The same operator $Q^w(t,x,D_t,D_x)$ can also be given by

$$Q^w(t,x,D_t,D_x)u = \sum_{j=0}^{l} D_s^{l-j}(A_j^w(\frac{t+s}{2},x,D_x)u(s,x))|_{s=t}. \quad (7)$$

For the proof of Theorem 1 we use the following.

Proposition 2. *Let* $p(t,x,\tau,\xi) = \tau^m + \sum_{j=1}^m a_j(t,x,\xi)\tau^{m-j}$ *with real* $a_j(t,x,\xi) \in S^j_{1,2/3}([0,T])$. *Suppose that we have*

$$p(t,x,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,x,\xi))$$

with real $\lambda_j(t,x,\xi) \in S^1_{1,2/3}([0,T])$ *satisfying, with some* $\delta > 0$,

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \geq \delta\langle\xi\rangle \qquad (t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \tag{8}$$

for any $j,k \in I$ *with* $j \neq k$. *Then we have the following:*

$$\sum_{j=0}^{m-1} \sum_{|\alpha|\leq m-1-j} \|D_t^j D_x^\alpha u(t,\cdot)\|_{-(m-1)/2} \leq$$
$$C(\sum_{j=0}^{m-1} \sum_{|\alpha|\leq m-1-j} \|D_t^j D_x^\alpha u(0,\cdot)\|^2_{-(m-1)/2} + \int_0^t \|P^w u(s,\cdot)\|_{-(m-1)/2}\, ds) \tag{9}$$

for any $u(t,x) \in C^\infty([0,T] \times \mathbb{R}^n)$ *that is rapidly decreasing with respect to* x.

In order to see that Proposition 2 implies Theorem 1, first we remark that an operator $A^w(t,x,D_x)$ with $a(t,x,\xi) \in S^s_{1,2/3}([0,T])$ has the order s, that is, we have $\|Au\|_\sigma \leq C\|u\|_{\sigma+s}$ (see Proposition 5.5 and the section 14 in Chapter 7 in Taylor's book[3)]) and that, for an operator $Q^w(t,x,D_t,D_x)$ with the symbol $q(t,x,\tau,\xi) = \sum_{j=0}^k a_j(t,x,\xi)\tau^{k-j}$ with $a_j(t,x,\xi) \in S^{s-k+j}_{1,2/3}([0,T])$, we have $\|Q(t,x,D_t,D_x)u\|_\sigma \leq C\sum_{j=0}^k \|D_t^{k-j}u\|_{j-k+s+\sigma}$ (see (7)).

Suppose that p satisfies the assumptions of Theorem 1. Set $q = \frac{-1}{8}\sum_{\mu,\nu=0}^n \partial_{\xi_\mu}\partial_{\xi_\nu}\partial_{x_\mu}\partial_{x_\nu}p$. Then we see that $q = \sum_{j=1}^m b_j(t,x,\xi)\tau^{m-j}$ with real symbols $b_j(t,x,\xi) \in S^{j-2/3}_{1,2/3}([0,T])$ and we have

$$\|P^w(t,x,D_t,D_x)u - (P + P_{sb} + Q)u\|_{-(m-1)/2} \leq C\sum_{j=0}^{m-1} \|D_t^j u\|_{m-j-1-(m-1)/2}. \tag{10}$$

Noting

$$\|Q^w u - Qu\|_{-(m-1)/2} \leq C\sum_{j=0}^{m-1} \|D_t^j u\|_{m-j-1-(m-1)/2}, \tag{11}$$

we have

$$\|(P^w - Q^w)u - (Pu + P_{sb}u)\|_{-(m-1)/2} \leq C\sum_{j=0}^{m-1} \|D_t^j u\|_{m-j-1-(m-1)/2}.$$

See (14.8) on page 60 in Taylor's book[3)] for the above two claims (10) and (11).

On the other hand, $p - q = \tau^m + \sum_{j=1}^m (a_j(t,x,\xi) - b_j(t,x,\xi))\tau^{m-j}$ has the factorization : when $|\xi|$ is large,

$$p - q = \prod(\tau - \tilde\lambda_j(t,x,\xi))$$

with real $\tilde\lambda_j(t,x,\xi)$ satisfying, with some $R_1 > 0$ and $\delta > 0$,

$$|\tilde\lambda_j(t,x,\xi) - \tilde\lambda_k(t,x,\xi)| \geq \delta\langle\xi\rangle \qquad (t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \text{ with } |\xi| \geq R_1$$

for any $j,k \in I$ with $j \neq k$. See the appendix for the proof of this claim.

The above $\tilde{\lambda}_j(t,x,\xi)$ $(j=1,\dots,m)$ can be extend as a real symbol in $S^1_{1,2/3}([0,T])$ satisfying (8). See the appendix for the proof of this claim. Using the extended $\tilde{\lambda}_j(t,x,\xi)$ $(j=1,\dots,m)$, we set $\tilde{p}(t,x,\tau,\xi)=\prod_{j=1}^m(\tau-\tilde{\lambda}_j(t,x,\xi))$. Then $\tilde{p}=p-q$ when $|\xi|$ is large. Hence we have

$$\|\tilde{P}^w u-(Pu+P_{sb}u)\|_{-(m-1)/2}\le C\sum_{j=0}^{m-1}\|D_t^j u\|_{m-1-j-(m-1)/2}.$$

While, applying Proposition 2 to $\tilde{p}(t,x,\tau,\xi)$, we obtain

$$\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(t,\cdot)\|_{-(m-1)/2}\le$$
$$C(\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(0,\cdot)\|_{-(m-1)/2}+\int_0^t\|\tilde{P}^w u(s,\cdot)\|_{-(m-1)/2}\,ds).$$

Then, noting that $\sum_{|\alpha|\le m-1-j}\|D_x^\alpha u\|_\sigma$ is equivalent to $\|u\|_{m-1-j+\sigma}$, we have

$$\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(t,\cdot)\|_{-(m-1)/2}\le$$
$$C(\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(0,\cdot)\|_{-(m-1)/2}$$
$$+\int_0^t(\|Pu(s,x)+P_{sb}u(s,\cdot)\|_{-(m-1)/2}+\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-1-j}\|D_t^jD_x^\alpha u(s,\cdot)\|_{-(m-1)/2})\,ds).$$

Then, by Gronwall's inequality we obtain the estimate (5). Hence Proposition 2 implies Theorem 1.

In the next section, we prove Proposition 2 by using the decomposition of P^w given in Proposition 3 that is proved in the section 3.

We denote by $L(k,s)$ with a non-negative integer k and a real s, the set of symbols $q(t,x,\tau,\xi)=\sum_{j=0}^k a_j(t,x,\xi)\tau^{k-j}$ with $a_j(t,x,\xi)\in S^{j-k+s}_{1,2/3}([0,T])$. $L(0,s)=S^s_{1,2/3}([0,T])$. For operators Q and R, we write $Q=R \mod L(k,s)$ when $Q-R$ can be given by $V^w(t,x,D_t,D_x)$ with some $v(t,x,\tau,\xi)\in L(k,s)$. While, for an operator Q, $Q\in S^k_{1,2/3}([0,T])$ means that Q can be given by $Q=A^w$ with $a(t,x,\xi)\in S^k_{1,2/3}([0,T])$.

For the operator $Q^w(t,x,D_t,D_x)$ whose symbol is $q(t,x,\tau,\xi)=\sum_{j=0}^k a_j(t,x,\xi)\tau^{k-j}\in L(k,s)$, the expression (7) implies that there exist $b_j(t,x,\xi)\in S^{j-k+s}_{1,2/3}([0,T])$ such that $b_j(t,x,\xi)-a_j(t,x,\xi)\in S^{j-1/3-k+s}_{1,2/3}([0,T])$ and

$$Q^w(t,x,D_t,D_x)u=\sum_{j=0}^k B_j^w(t,x,D_x)D_t^{k-j}u.$$

We recall some properties of the Weyl calculus. See (14.24) on page 62 in Taylor's book[3)]. We define the Poisson bracket $\{q,r\}$ of $q(t,x,\tau,\xi)\in L(k,s_1)$ and $r(t,x,\tau,\xi)\in L(l,s_2)$, by

$$\{q,r\}=\sum_{\mu=0}^n(\partial_{\xi_\mu}q\partial_{x_\mu}r-\partial_{x_\mu}q_1\partial_{\xi_\mu}q_2),$$

and $(q,r)_2$ by

$$(q,r)_2=\frac{-1}{4}\sum_{\mu,\nu=0}^n(q^{(\mu,\nu)}r_{(\mu,\nu)}+q_{(\mu,\nu)}r^{(\mu,\nu)}-2q^{(\nu)}_{(\mu)}r^{(\mu)}_{(\nu)}) \tag{12}$$

where $q^{(\mu,\nu)} = \partial_{\xi_\mu}\partial_{\xi_\nu} q$, $q_{(\mu,\nu)} = \partial_{x_\mu}\partial_{x_\nu} q$ and $q^{(\nu)}_{(\mu)} = \partial_{x_\mu}\partial_{\xi_\nu} q$ with $x_0 = t$ and $\xi_0 = \tau$. We use also the notation $q^{(\mu)} = \partial_{\xi_\mu} q$, $q_{(\mu)} = \partial_{x_\mu} q$.

Here we note $\{q, r\} = -\{r, q\}$ and $(q, r)_2 = (r, q)_2$.

We set

$$d_0 = qr,\ d_1 = \frac{1}{2i}\{q, r\},\ d_2 = \frac{1}{2}(q, r)_2.$$

We remark that $d_j \in L(k + l, s_1 + s_2 - j/3)$ $(j = 0, 1, 2)$. We have, for $j_0 = 0, 1, 2$,

$$Q^w R^w = \sum_{j=0}^{j_0} D_j^w \quad (\mathrm{mod}\ L(k + l, s_1 + s_2 - (j_0 + 1)/3)) \tag{13}$$

and

$$Q^w R^w - R^w Q^w = 2D_1^w \quad (\mathrm{mod}\ L(k + l, s_1 + s_2 - 1))$$

Furthermore when $q(t, x, \tau, \xi) - \tau^k \in L(k - 1, s_1)$ or $r(t, x, \tau, \xi) - \tau^l \in L(l - 1, s_2)$, we have for $j_0 = 0, 1, 2$,

$$Q^w R^w = \sum_{j=0}^{j_0} D_j^w \quad (\mathrm{mod}\ L(k + l - 1, s_1 + s_2 - (j_0 + 1)/3)) \tag{14}$$

and

$$Q^w R^w - R^w Q^w = 2D_1^w \quad (\mathrm{mod}\ L(k + l - 1, s_1 + s_2 - 1)) \tag{15}$$

In the following we use the following notations. We denote the inner product in the space of square integrable functions on $\mathbb{R}^n$, by $(\cdot, \cdot)$, that is,

$$(v, w) = \int_{\mathbb{R}^n} v(x)\overline{w(x)}\, dx$$

and its norm by $\|v(\cdot)\| = (v, v)^{1/2}$.

We use C or C with some suffix in order to denote a positive constant which may be different line by line.

2. Proof of Proposition 2.

In this section, because we use only the Weyl calculus, the operator define by (6) or (7) is denoted by A or Q instead of A^w or Q^w.

Suppose that the assumption of Proposition 2 is fulfilled. Recall $I = \{1, 2, \ldots, m\}$. For $j \in I$ we set

$$p_j(t, x, \tau, \xi) = \sum_{k \in I\setminus\{j\}} (\tau - \lambda_k(t, x, \tau, \xi)).$$

We have the following.

Proposition 3. *We set, for $j \in I$,*

$$e_j^0(t, x, \xi) = \sum_{k \in I\setminus\{j\}} \frac{-1}{2i} \frac{\{\tau - \lambda_j, \tau - \lambda_k\}}{\lambda_j - \lambda_k}.$$

There exist a real symbol $e_j^1(t, x, \xi) \in S^{1/3}_{1,2/3}([0, T])$ and a symbol $r_j(t, x, \tau, \xi) \in L(m - 1, m - 1/3)$ such that

$$(D_t - \Lambda_j + E_j^0 + E_j^1)(P_j + R_j) - P \in L(m - 1, m - 1). \tag{16}$$

For the moment admitting the validity of the above proposition, we draw the estimates (9). In the following, we denote by u a C^∞ function on $[0,T]\times\mathbb{R}^n$, which is rapidly decreasing with respect to x.

Set

$$w_j(t,x,\xi)=\prod_{k\in I\setminus\{j\}}|\lambda_j-\lambda_k|^{-1/2}.$$

We see from (8) that $w_j\in S^{-(m-1)/2}_{1,2/3}([0,T])$ and $w_j(t,x,\xi)\geq C(1+|\xi|)^{-(m-1)/2}$. We have

$$\{\tau-\lambda_j,w_j\}=w_j(-\sum_{k\in I\setminus\{j\}}\frac{1}{2(\lambda_j(t,x,\xi)-\lambda_k(t,x,\xi))}\{\tau-\lambda_j,\lambda_j-\lambda_k\}).$$

Then $\frac{1}{i}\{\tau-\lambda_j,w_j\}=w_je_j^0$. Setting $g=w_je_j^0$, we get from (15)

$$(D_t-\Lambda_j)W_j-W_j(D_t-\Lambda_j)=G\pmod{S^{-(m-1)/2}_{1,2/3}}.$$

While with $\tilde e_j^1\in S^{1/3}_{1,2/3}$ given by $\tilde e_j^1=-\frac{1}{2i}\{w_j,e_j^0\}/w_j$ which is real-valued, we have

$$W_jE_j^0=G-\tilde E_j^1W_j\pmod{S^{-(m-1)/2}_{1,2/3}([0,T])}.$$

Then we have

$$(D_t-\Lambda_j)W_j-W_j(D_t-\Lambda_j)-W_jE_j^0-\tilde E_j^1W_j\in S^{-(m-1)/2}_{1,2/3},$$

which and $E_j^1W_j=W_jE_j^1$ (mod $S^{-(m-1)/2}_{1,2/3}([0,T])$), where $e_j^1\in S^{1/3}_{1,2/3}([0,T])$, imply that

$$W_j(D_t-\Lambda_j+E_j^0+E_j^1)-(D_t-\Lambda_j+E_j^1+\tilde E_j^1)W_j\in S^{-(m-1)/2}_{1,2/3}. \tag{17}$$

Then we see from (16) and (17) that there exists $v_j(t,x,\tau,\xi)\in L(m-1,(m-1)/2)$ such that

$$(D_t-\Lambda_j+E_j^1+\tilde E_j^1)W_j(P_j+R_j)=W_jP+V_j. \tag{18}$$

Since $-\lambda_j(t,x,\xi)+e_j^1(t,x,\xi)+\tilde e_j^1(t,x,\xi)$ is real-valued, then $-\Lambda_j(t,x,D_x)+E_j^1(t,x,D_x)+\tilde E_j^1(t,x,D_x)$ is formally selfadjoint with respect to the inner product (u,v). Then we see that

$$\frac{d}{dt}\|u\|^2=2\Re(iD_tu(t,\cdot),u(t,\cdot))=2\Re(i(D_t-\Lambda_j+E_j^1+\tilde E_j^1)u(t,\cdot),u(t,\cdot)).$$

Hence $\frac{d}{dt}\|W_ju\|^2=2\Re(i(D_t-\Lambda_j+E_j^1+\tilde E_j^1)W_ju(t,\cdot),W_ju(t,\cdot))$. Then

$$\frac{d}{dt}\|W_ju\|^2\leq 2\|(D_t-\Lambda_j+E_j^1+\tilde E_j^1)W_ju\|\|W_ju\|.$$

Therefore, noting $w_j\in S^{-(m-1)/2}_{1,2/3}([0,T])$, we obtain from (18)

$$\frac{d}{dt}\|W_j(P_j+R_j)u\|^2\leq C(\|Pu\|_{-(m-1)/2}+\sum_{k=0}^{m-1}\|D_t^ku\|_{-k+(m-1)/2})\|W_j(P_j+R_j)u\|,$$

from which we obtain

$$\begin{aligned}\|W_j(P_j+R_j)u(t,\cdot)\|&\leq\|W_j(P_j+R_j)u(0,\cdot)\|\\&\quad+C\int_0^t(\|Pu(s,\cdot)\|_{-(m-1)/2}+\sum_{k=0}^{m-1}\|D_t^ku(s,\cdot)\|_{-k+(m-1)/2})\,ds.\end{aligned}$$

See for example Lemma 23.1.1 in the book of Hömander[4)].

Since $w_j \in S^{-(m-1)/2}_{1,2/3}([0,T])$ and $w_j \geq C(1+|\xi|)^{-(m-1)/2}$, we have $\|u\|_{-(m-1)/2} \leq C(\|W_j u\| + \|u\|_{-(m-1)/2-1})$. Hence we see

$$\begin{aligned}\|(P_j+R_j)u(t,\cdot)\|_{-(m-1)/2} \leq C(&\sum_{k=0}^{m-1}\|D_t^k u(t,\cdot)\|_{-k+(m-1)/2-1}\\ &+\sum_{k=0}^{m-1}\|D_t^k u(0,\cdot)\|_{-k+(m-1)/2}\\ &+\int_0^t(\|Pu(s,\cdot)\|_{-(m-1)/2}+\sum_{k=0}^{m-1}\|D_t^k u(s,\cdot)\|_{-k+(m-1)/2})\,ds. \quad (19)\end{aligned}$$

Now we draw the estimate of the norm of $D_t^k u$ from the estimate of $\|(P_j+R_j)u\|_{-(m-1)/2}$ ($j = 1,\ldots,m$). First we note that $r_j \in L(m-2, m-1-1/3)$ implies that

$$\|R_j u\|_{-(m-1)/2} \leq C\sum_{k=0}^{m-2}\|D_t^k u\|_{-k+(m-1)/2-1/3} \quad (20)$$

Note that we have

$$p_j(t,x,\tau,\xi) = \tau^{m-1} + \sum_{k=2}^{m} a_{jk}(t,x,\xi)\tau^{m-k}\langle\xi\rangle^{k-1}$$

with $a_{jk}(t,x,\xi) \in S^0_{1,2/3}([0,T])$.

Let A be a m by m matrix whose (j,k) element is $a_{jk}(t,x,\xi)$. Here we set $a_{j1}(t,x,\xi) = 1$. We remark that

$$p_j(t,x,\tau,\xi)/\langle\xi\rangle^{m-1} = \sum_{k=1}^{m} a_{jk}(t,x,\xi)\big(\frac{\tau}{\langle\xi\rangle}\big)^{m-k}$$

Then we have $|\det A| \geq C$ with $C > 0$. Indeed, since

$$p_j(t,x,\lambda_l(t,x,\xi),\xi) = \begin{cases}\prod_{k\in I\setminus\{j\}}(\lambda_j(t,x,\xi)-\lambda_k(t,x,\xi)) & l=j\\ 0 & \text{otherwise,}\end{cases}$$

and

$$p_j(t,x,\lambda_l(t,x,\xi),\xi)/\langle\xi\rangle^{m-1} = \sum_{k=1}^{m} a_{jk}(t,x,\xi)\big(\frac{\lambda_l(t,x,\xi)}{\langle\xi\rangle}\big)^{m-k},$$

then we obtain the desired estimate $|\det A| \geq C$ from (8) and the Vandermonde determinant.

Hence each element of A^{-1} belongs to $S^0_{1,2/3}([0,T])$. Then there exists an m by m matrix $L \in S^0_{1,2/3}([0,T])$ satisfying $L(t,x,D_x)A(t,x,D_x) - I \in S^{-1}_{1,2/3}([0,T])$. Noting that

$$\|P_j u - \sum_{k=1}^{m} A_{jk}(t,x,D_x)D_t^{m-k}\langle D_x\rangle^{k-1}u\|_{-(m-1)/2} \leq C\sum_{l=0}^{m-2}\|D_t^l u\|_{-l+(m-1)/2-1/3},$$

we see that, for $k \in I$,

$$\|\sum_{j=1}^{m} L_{kj}(t,x,D_x)P_j u - D_t^{m-k}\langle D_x\rangle^{k-1}u\|_{-(m-1)/2} \leq C\sum_{l=0}^{m-1}\|D_t^l u\|_{-l+(m-1)/2-1/3},$$

where L_{kl} is the (k, j) element of the L. Then we obtain from the above estimate and (20)

$$\sum_{k=0}^{m-1} \|D_t^{m-1-k} u(t,\cdot)\|_{k-(m-1)/2} \le C(\sum_{j=1}^{m} \|(P_j + R_j)u\|_{-(m-1)/2} + \sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2-1}). \quad (21)$$

Here we used the property of Sobolev norm: $\|u\|_{\sigma-1/3} \le \varepsilon \|u\|_\sigma + C_\varepsilon \|u\|_{\sigma-1}$ for any $\varepsilon > 0$ with some C_ε.

It follows from (19) and (21)

$$\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2} \le C(\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2-1} + \sum_{k=0}^{m-1} \|D_t u(0,\cdot)\|_{-k+(m-1)/2}$$
$$+ \int_0^t (\|Pu(s,\cdot)\|_{-(m-1)/2} + \sum_{k=0}^{m-1} \|D_t^k u(s,\cdot)\|_{-k+(m-1)/2})\, ds. \quad (22)$$

Note that for $0 \le k \le m-2$

$$\frac{d}{dt} \|D_t^k u(t,\cdot)\|^2_{-k+(m-1)/2-1} \le 2\|D_t^k u(t,\cdot)\|_{-k+(m-1)/2-1} \|D_t^{k+1} u(t,\cdot)\|_{-k+(m-1)/2-1}.$$

and

$$\frac{d}{dt} \|D_t^{m-1} u(t,\cdot)\|^2_{-(m-1)/2-1}$$
$$\le 2\|D_t^{m-1} u(t,\cdot)\|_{-(m-1)/2-1} (\|(Pu(t,\cdot)\|_{-(m-1)/2-1} + \sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{(m-k-(m-1)/2-1}). \quad (23)$$

Then we have

$$\frac{d}{dt} (\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|^2_{-k+(m-1)/2-1})$$
$$\le C(\|Pu\|_{-(m-1)/2-1} + \sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2}) (\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|^2_{k-(m-1)/2-1})^{1/2},$$

from which we obtain

$$\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2-1} \le C(\sum_{k=0}^{m-1} \|D_t^k u(0,\cdot)\|_{-k+(m-1)/2-1}$$
$$+ \int_0^t (\|Pu(s,\cdot)\|_{-(m-1)/2-1} + \sum_{k=0}^{m-1} \|D_t^k u(s,\cdot)\|_{-k+(m-1)/2})\, ds).$$

Hence from the above estimate and (22), we obtain

$$\sum_{k=0}^{m-1} \|D_t^k u(t,\cdot)\|_{-k+(m-1)/2} \le C(\sum_{k=0}^{m-1} \|D_t^k u(0,\cdot)\|_{-k+(m-1)/2}$$
$$+ \int_0^t (\|Pu(s,\cdot)\|_{-(m-1)/2} + \sum_{k=0}^{m-1} \|D_t^k u(s,\cdot)\|_{-k+(m-1)/2})\, ds. \quad (24)$$

Then we obtain from (24) the desired estimate (9), by using Gronwall's inequality.

3. Proof of Proposition 3.

In this section also, we use only the Weyl calculus. Then the operator define by (6) or (7) is denoted by A or Q instead of A^w or Q^w.

First we show two lemmas on symbols.

Recall $I = \{1, 2, \ldots, m\}$. For $j, k \in I$ with $j \neq k$, we set

$$p_j = \prod_{l \in I \setminus \{j\}} (\tau - \lambda_l(t, x, \xi)), \; p_{kj} = \prod_{l \in I \setminus \{k,j\}} (\tau - \lambda_l(t, x, \xi)).$$

When $m = 2$, we set $p_{jk} = 1$.

Lemma 4. *For $j \in I$, we set*

$$a_j^0 = - \sum_{k \in I \setminus \{j\}} \frac{1}{2i} \frac{\{\tau - \lambda_j, \lambda_k - \lambda_j\}}{\lambda_k - \lambda_j}$$

and for $k \in I \setminus \{j\}$,

$$b_{jk}^0 = \frac{1}{2i} \frac{\{\tau - \lambda_j, \lambda_k - \lambda_j\}}{\lambda_k - \lambda_j}.$$

We set also

$$r_j^0 = \sum_{k \in I \setminus \{j\}} b_{jk} p_{jk}.$$

Then we have

$$\frac{1}{2i} \{\tau - \lambda_j, p_j\} + a_j^0 p_j + (\tau - \lambda_j) r_j^0 = 0.$$

Proof. Indeed, by the definition of the Poisson bracket, we see that $\{\tau - \lambda_j, p_j\} = \sum_{k \in I \setminus \{j\}} \{\tau - \lambda_j, \tau - \lambda_k\} p_{jk}$. Since $1 = \frac{(\tau - \lambda_j) - (\tau - \lambda_k)}{\lambda_k - \lambda_j}$, noting $(\tau - \lambda_j) p_{jk} = p_k$ and $(\tau - \lambda_k) p_{jk} = p_j$, we get

$$p_{jk} = \frac{p_k - p_j}{\lambda_k - \lambda_j}. \tag{25}$$

Since $\{\tau - \lambda_j, \tau - \lambda_j\} = 0$ and $(\tau - \lambda_k) - (\tau - \lambda_j) = \lambda_j - \lambda_k$, we see that $\{\tau - \lambda_j, \tau - \lambda_k\} = \{\tau - \lambda_j, \lambda_j - \lambda_k\}$. Hence

$$\{\tau - \lambda_j, p_j\} = \sum_{k \in I \setminus \{j\}} \{\tau - \lambda_j, \lambda_j - \lambda_k\} \frac{p_k - p_j}{\lambda_k - \lambda_j}.$$

Then, noting $(\tau - \lambda_j) p_{jk} = p_k$, we obtain

$$\frac{1}{2i} \{\tau - \lambda_j, p_j\} + a_j^0 p_j + (\tau - \lambda_j) r_j^0 = 0.$$

□

Lemma 5. *There exist real symbols $a_j^{1,h}, b_{jk}^{1,h} \in S_{1,2/3}^{1/3}([0, T])$ $(h = 1, 2, 3, 4)$ such that*

$$(\tau - \lambda_j, p_j)_2 = a_j^{1,1} p_j + \sum_{k \in I \setminus \{j\}} b_{jk}^{1,1} p_k \tag{26}$$

$$\frac{1}{2i} \{a_j^0, p_j\} = a_j^{1,2} p_j + \sum_{k \in I \setminus \{j\}} b_{jk}^{1,2} p_k \tag{27}$$

$$\frac{1}{2i} \{\tau - \lambda_j, r_j^0\} = a_j^{1,3} p_j + \sum_{k \in I \setminus \{j\}} b_{jk}^{1,3} p_k \tag{28}$$

$$a_j^0 r_j^0 = a_j^{1,4} p_j + \sum_{k \in I \setminus \{j\}} b_{jk}^{1,4} p_k \tag{29}$$

where a_j^0 and r_j^0 are symbols defined in Lemma 4.

Proof. For three distinct elements j, k, l in I, we set

$$p_{jkl} = \prod_{h \in I \setminus \{j,k,l\}} (\tau - \lambda_h).$$

When $m = 3$, we set $p_{jkl} = 1$.

Lagrange's interpolation formula implies that, for three distinct elements $j, k, l \in I$,

$$1 = \frac{(\tau - \lambda_j)(\tau - \lambda_k)}{(\lambda_l - \lambda_j)(\lambda_l - \lambda_k)} + \frac{(\tau - \lambda_k)(\tau - \lambda_l)}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)} + \frac{(\tau - \lambda_l)(\tau - \lambda_j)}{(\lambda_k - \lambda_l)(\lambda_k - \lambda_j)}$$

Then we have

$$p_{jkl} = \frac{p_l}{(\lambda_l - \lambda_j)(\lambda_l - \lambda_k)} + \frac{p_j}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)} + \frac{p_k}{(\lambda_k - \lambda_l)(\lambda_k - \lambda_j)} \tag{30}$$

Note that we have for $\mu, \nu \in \{0, 1, 2, \ldots, n\}$

$$p_{j(\mu,\nu)} = \sum_{k \in I \setminus \{j\}} (\tau - \lambda_k)_{(\mu,\nu)} p_{jk} + \sum_{k,l \in I \setminus \{j\}, k \neq l} (\tau - \lambda_k)_{(\mu)} (\tau - \lambda_l)_{(\nu)} p_{jkl}.$$

Then we see from (25) and (30) that

$$\begin{aligned}
(\tau - \lambda_j)^{(\mu,\nu)} p_{j(\mu,\nu)} = &\sum_{k \in I \setminus \{j\}} (\tau - \lambda_j)^{(\mu,\nu)} (\tau - \lambda_k)_{(\mu,\nu)} \frac{p_k - p_j}{\lambda_k - \lambda_j} \\
&+ \sum_{k,l \in I \setminus \{j\}, k \neq l} (\tau - \lambda_j)^{(\mu,\nu)} (\tau - \lambda_k)_{(\mu)} (\tau - \lambda_l)_{(\nu)} \\
&\times \left(\frac{p_l}{(\lambda_l - \lambda_j)(\lambda_l - \lambda_k)} + \frac{p_j}{(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)} + \frac{p_k}{(\lambda_k - \lambda_l)(\lambda_k - \lambda_j)} \right).
\end{aligned} \tag{31}$$

Since λ_k $(k \in I)$ is real-valued, we see that there exist real symbols $a_k \in S^{1/3}_{1,2/3}([0, T])$ so that we have

$$(\tau - \lambda_j)^{(\mu,\nu)} p_{j(\mu,\nu)} = \sum_{k \in I} a_k p_k.$$

Since this argument can be applied to other terms of $(\tau - \lambda_j, p_j)_2$ (see (12)), we see that the assertion for (26) is valid. Noting that $\frac{a^0_j}{i} \in S^{2/3}_{1,2/3}([0, T])$ is real valued, we see that the assertion for (27) follows from

$$\begin{aligned}
\frac{1}{2i}\{a^0_j, p_j\} &= \sum_{k \in I \setminus \{j\}} \frac{1}{2i}\{a^0_j, \tau - \lambda_k\} p_{jk} \\
&= \sum_{k \in I \setminus \{j\}} \frac{1}{2i}\{a^0_j, \tau - \lambda_k\} \frac{p_k - p_j}{\lambda_k - \lambda_j}.
\end{aligned}$$

Since

$$\frac{1}{2i}\{\tau - \lambda_j, b^0_{jk} p_{jk}\} = \frac{1}{2i}\{\tau - \lambda_j, b^0_{jk}\} p_{jk} + \frac{b^0_{jk}}{2i}\{\tau - \lambda_j, \tau - \lambda_l\} p_{jkl},$$

noting that $\frac{b^0_{jk}}{i} \in S^{2/3}_{1,2/3}([0, T])$ is real valued, we see from (25) and (30), the assertion for (28) is valid. Finally, noting that $a^0_j r^0_j$ is real-valued, we see from the definition of a^0_j and r^0_j and from (25) that the assertion for (29) is valid. $\square$

We see from (14) that

$$(D_t-\Lambda_j)P_j = P + Q_{j,1} + Q_{j,2} \pmod{L(m-1,m-1)}$$

with $q_{j1} = \frac{1}{2i}\{\tau-\lambda_j, p_j\}$ and $q_{j,2} = \frac{1}{2}(\tau-\lambda_j, p_j)_2$.

Setting $h_1(t,x,\tau,\xi) = \frac{1}{2i}\{a_j^0, p_j\} + \frac{1}{2i}\{\tau-\lambda_j, r_j^0\}$, we see from Lemma 4 and (14) that

$$A_j^0 P_j + (D_t-\Lambda_j)R_j^0 + Q_{j1} = H_1 \pmod{L(m-2,m-1)}.$$

Then, setting $h_2(t,x,\tau,\xi) = h_1(t,x,\tau,\xi) + q_{j2} + a_j^o r_j^0$, we see that

$$(D_t-\Lambda_j + A_j^0)(P_j + R_j^0) = P + H_2 \pmod{L(m-1,m-1)}.$$

Since

$$h_2(t,x,\tau,\xi) = \frac{1}{2i}\{a_j^0, p_j\} + \frac{1}{2i}\{\tau-\lambda_j, r_j^0\} + \frac{1}{2}(\tau-\lambda_j, p_j)_2 + a_j^o r_j^0,$$

Lemma 5 shows the following expression of h_2 with real symbols $a_j^1, b_{jk}^1 \in S_{1,2/3}^{1/3}([0,T])$

$$h_2(t,x,\tau,\xi) = -a_j^1 p_j - \sum_{k\in I\setminus\{j\}} b_{jk}^1 p_k$$

With $r_j^1 = \sum_{k\in I\setminus\{j\}} b_{jk}^1 p_{jk} \in L(m-2, m-2+1/3)$ we have

$$h_2(t,x,\tau,\xi) = -a_j^1 p_j - (\tau-\lambda_j) r_j^1$$

Then we see that

$$H_2 = -A_j^0 P_j - (D_t-\Lambda_j)R_j^1 \pmod{L(m-2,m-1)}.$$

Then we obtain

$$(D_t-\Lambda_j + A_j^0 + A_j^1)(P_j + R_j^0 + R_j^1) = P \pmod{L(m-1,m-1)}.$$

Since $a_j^1(t,x,\xi)$ is real-valued, Proposition 3 is proved. □

4. Appendix.

We show the validity of two claims stated in Introduction. Assume that the symbol $p(t,x,\tau,\xi) = \tau^m + \sum_{j=1}^m a_j(t,x,\xi)\tau^{m-j}$ with real $a_j(t,x,\xi) \in S_{1,2/3}^j([0,T])$ has the following factorization when $|\xi| \ge R_0$ with some $R_0 > 0$.

$$p(t,x,\tau,\xi) = \prod_{j=1}^m (\tau - \lambda_j(t,x,\xi)) \tag{32}$$

with real $\lambda_j(t,x,\xi)$ satisfying, with some $\delta > 0$,

$$|\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi)| \ge \delta\langle\xi\rangle \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \text{ with } |\xi| \ge R_0 \tag{33}$$

for any $j,k \in I$ with $j \ne k$.

The first claim is the following. For any $q(t,x,\tau,\xi) = \sum_{j=1}^m b_j(t,x,\xi)\tau^{m-j}$ with real $b_j(t,x,\xi) \in S_{1,2/3}^{j-2/3}([0,T])$, $p(t,x,\tau,\xi) + q(t,x,\tau,\xi)$ has the factorization similar to (32) and (33) when $|\xi| \ge R_1$ with some $R_1 > 0$. Indeed, when $|\xi| \ge R_0$, set $p_j = \prod_{l\in I\setminus\{j\}}(\tau - \lambda_k(t,x,\xi))$. Then, by Lagrange's interpolation formula, we have

$$q(t,x,\tau,\xi) = \sum_{j=1}^m \frac{d_j}{p_j}$$

with

$$d_j = \frac{q(t,x,\lambda_j(t,x,\xi),\xi)}{p_j(t,x,\lambda_j(t,x,\xi),\xi)}.$$

Since $p_j(t,x,\lambda_j(t,x,\xi),\xi) = \prod_{l\in I\setminus\{j\}}(\lambda_j(t,x,\xi)-\lambda_k(t,x,\xi))$, we see from (33)

$$|p_j(t,x,\lambda_j(t,x,\xi),\xi)| \geq C\langle\xi\rangle^{m-1} \quad (|\xi| \geq R_0).$$

Note that the estimates $|a_j| \leq C\langle\xi\rangle^j$ $(j=1,\ldots,m)$ imply $|\lambda_j| \leq C\langle\xi\rangle$ $(j=1,\ldots,m)$. Then from $b_j(t,x,\xi) \in S^{j-2/3}_{1,2/3}([0,T])$ we obtain $|q(t,x,\lambda_j(t,x,\xi),\xi)| \leq C\langle\xi\rangle^{m-1-2/3}$. Hence $|d_j| \leq C\langle\xi\rangle^{-2/3}$. Then d_j is small when $|\xi|$ is large. Since $b_j(t,x,\xi)$ is real-valued, d_j is real-valued. From $p+q = p(1+\sum_{j=1}^m \frac{d_j}{\tau-\lambda_j})$ and from (33), by considering the change of sign, follows that there exists a positive R_1 such that, when $|\xi| \geq R_1$, we have the factorization of $p+q$ similar to (32) and (33). Then we see that the first claim is valid.

The second claim is the following. Each $\lambda_j(t,x,\xi)$ $(j=1,\ldots,m)$ appearing in (32) has an real extention $\tilde{\lambda}_j \in S^1_{1,2/3}([0,T])$ satisfying, with some $\delta > 0$,

$$|\tilde{\lambda}_j(t,x,\xi) - \tilde{\lambda}_k(t,x,\xi)| \geq \delta\langle\xi\rangle \quad (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \tag{34}$$

for any $j,k \in I$ with $j \neq k$.

We show this claim by using Nuij's theorem[5)]. We assume $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ when $|\xi| \geq R_0$. Let a non-negative function $\chi(s) \in C^\infty(\mathbb{R})$ satisfy

$$\chi(s) = \begin{cases} 1 & |s| \leq 3/2 \\ 0 & |s| \geq 2. \end{cases}$$

Note that by the implicit function theorem and (33), we see $(1-\chi(|\xi|/R_0))\lambda_j(t,x,\xi) \in S^1_{1,2/3}([0,T])$. Set $\tilde{p} = \prod_{j=1}^m(\tau - (1-\chi(|\xi|/R_0)\lambda_j(t,x,\xi))$. The polynomial $\tilde{p}$ has only real roots that are distinct when $|\xi| \geq 2R_0$. Thanks to Nuij's theorem, we see that

$$\tilde{\tilde{p}} = (1+\chi(\frac{|\xi|}{2R_0})\partial_\tau)^{m-1}\tilde{p}$$

has only simple real roots $\tilde{\lambda}_j$ $(j=1,\ldots,m)$. We assume $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots < \tilde{\lambda}_m$. Since $\tilde{\tilde{p}} = p$ when $|\xi| \geq 4R_0$, we see that $\lambda_j = \tilde{\lambda}_j$ there. Then (34) is verified when $|\xi| \geq 4R_0$. When $|\xi| \leq 4R_0$, the coefficients of $\tilde{p}$ are bounded. Then as a polynomial in τ, the set $\{\tilde{p}(t,x,\tau,\xi) \mid (t,x,\xi) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$ with $|\xi| \leq 4R_0$ $\}$ is a bounded set. For any $\tilde{p}$ belonging to its closure that has also only real roots, $\tilde{\tilde{p}}$ has only simple real roots. Then we see that, even if $|\xi| \leq 4R_0$, (34) is valid. Hence the claim is verified.

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